Some Solved Problems for Week Two

1.4#1 We take as true the following propositions:

> If Dracula seizes power, then democracy is lost.
> If democracy is lost, then the use of food additives increases.
> If the use of food additives increases, then mutations can be expected.
> Dracula seizes power.

Represent each proposition by a sentential form, then show that repeated application of modus ponens proves the truth of "mutations can be expected."

Solution: First we need a "dictionary" of variables and assignments:

D is "Dracula seizes power."

L is "democracy is lost."

U is "the use of food additives increases."

M is "mutations can be expected."

A formal deduction of "mutations can be expected" from the given premises might look like this:

1	D⇒L	Premise
2	L⇒U	Premise
3	U⇒M	Premise
4	D	Premise
5	L	Modus Ponens, lines 1, 4.
6	U	Modus Ponens, lines 2, 5.
7	M	Modus Ponens, lines 3, 6.

1.4#2 Use ordinary English to write sentences representing the converse, the contrapositive, and the negation of "If all cats meow, then some dogs bark."

Solution: This might not be obvious until we cover quantification. Neverthless, here are the solutions. The converse: "If some dogs bark, then all cats meow." The contrapositive: "If no dogs bark, then some cats don't meow." The negation: "All cats meow, but no dogs bark."

1.4#4 Prove that the sum of two odd numbers or two even numbers is even, and that the sum of an even number and an odd number is odd.

Solution: I'll do the last, leaving the others to you. Let m and n be integers, with m even and n odd. By definition, there are integers j and k such that $m = 2j$ and $n = 2k + 1$. It follows that $m + n = 2j + (2k + 1) = 2(j + k) + 1$, an odd number. □

1.5#3 Write sentential forms logically equivalent to $P \Rightarrow (Q \Rightarrow R)$ and $P \vee Q$ in which the symbol "$\Rightarrow$" does not occur.

Solution: Using the definition of $P \Rightarrow Q$ as an abbreviation for $\neg(P \wedge \neg Q)$ a couple of times, using double negation once, and using associativity, we get

$$P \Rightarrow (Q \Rightarrow R) \equiv \neg(P \wedge Q \wedge \neg R).$$

Note that this allows us to do our business using only negation and conjunction.

1.5#5 We write $P \downarrow Q$ as an abbreviation for $\neg(P \vee Q)$ (the down arrow is called the *dagger*). Represent each of the sentential forms $\neg P$, $P \vee Q$, $P \wedge Q$, and $P \Rightarrow Q$ by forms using only the dagger.

Solution: The process was discussed in class; here are the results.

(a) $\neg P \equiv P \downarrow P$

(b) $P \vee Q \equiv (P \downarrow Q) \downarrow (P \downarrow Q)$

(c) $P \wedge Q \equiv (P \downarrow P) \downarrow (Q \downarrow Q)$

(d) $P \Rightarrow Q \equiv ((P \downarrow P) \downarrow Q) \downarrow ((P \downarrow P) \downarrow Q)$

2.1#2 (a) $A = \{0, 2, 4, 6, 8, \ldots\} = \{2n | n \in \mathbf{Z}_{nn}\}$.

(c) $C = \{1, 5, 9, 13, \ldots\} = \{4n - 3 | n \in \mathbf{N}\}$.

(e) $E = \{lemon, 1, 2, 3, 4, 5, \ldots\} = \{x | x = lemon \text{ or } x \in \mathbf{N}\}$.

2.1#3 Describe each by listing all elements.

(a) $\{x \in \mathbf{N} | x \text{ is prime and } x^2 < 30\} = \{2, 3, 5\}$

(d) $\{x \in \mathbf{Z} | \text{ There is an element } y \in \mathbf{N} \text{ such that } x^2 + y^2 \leq 25\} = \{0, \pm 1, \pm 2, \pm 3, \pm 4\}$

1. Label as true or false:

(b) $\{1, 2, 3\} = \{2, 1, 3, 3, 2\}$ is true, since order doesn't count and multiples are meaningless.

(d) $\{5\} \in \{2, 5\}$ is false, since the set on the right-hand side contains no sets as elements.

(e) $\emptyset \in \{1, 2\}$ is false, since we don't see $\emptyset$ in the list of elements on the right-hand side.

(j) $\emptyset \in \emptyset$ is false, since $\emptyset$ contains no elements.

2.1#7 In each part, give an example of sets A, B, C that simultaneously satisfy the stated conditions. Note that there is nothing unique about these solutions!

(a) $A \in B, B \in C, A \notin C$. How about $B = \{A\}$ and $C = \{B\} = \{\{A\}\}$.

(b) $A \in B, B \in C, A \in C$. Try $B = \{A\}$ and $C = \{A, B\} = \{A, \{A\}\}$.